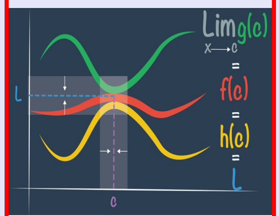
Calculus I

Lecture 13



Feb 19-8:47 AM

Prove $\lim_{x\to 10}\left(3-\frac{4}{5}x\right)=-5$ using

ε and δ definition of a limit.

$f(x)=3-\frac{4}{5}x$, $a=10$, $L=-5$ ✓

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$\left|3-\frac{4}{5}x-(-5)\right|<\varepsilon$ $\qquad$ $|x-10|<\delta$

$\left|-\frac{4}{5}x+8\right|<\varepsilon$ $\qquad$ $\left|\frac{-4}{5}\right||x-10|<\varepsilon$

$\left|\frac{-4}{5}\left(x-\frac{5}{4}\cdot 8\right)\right|<\varepsilon$ $\qquad$ $\frac{4}{5}|x-10|<\varepsilon$

$\left|\frac{-4}{5}(x-10)\right|<\varepsilon$ $\qquad$ $|x-10|<\frac{5\varepsilon}{4}$

$\delta=\frac{5\varepsilon}{4}$

Sep 17-7:21 AM

Prove $\lim_{x \to 2} (x^2 - 4x + 5) = 1$ using ε and δ definition of a limit.

$f(x) = x^2 - 4x + 5$, $a = 2$, $L = 1$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - 2| < \delta$

$|x^2 - 4x + 5 - 1| < \varepsilon$ = $|x - 2| < \delta$

$|x^2 - 4x + 4| < \varepsilon$ → take square-root

$|(x-2)(x-2)| < \varepsilon$ $|x - 2| < \sqrt{\varepsilon}$

$|(x-2)^2| < \varepsilon$ $\delta = \sqrt{\varepsilon}$

Sep 17-7:34 AM

Prove $\lim_{x \to 2} (x^3 + x) = 10$

$f(x) = x^3 + x$ $a = 2$ $L = 10$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^3 + x - 10| < \varepsilon$ = $|x - 2| < \delta$

Synthetic Div.

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 1 & -10 \\ & & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$|(x-2)(x^2 + 2x + 5)| < \varepsilon$

$\underbrace{|x^2 + 2x + 5|}_{\text{Bound}} \underbrace{|x - 2|}_{\text{Keep}} < \varepsilon$

If $|x^2 + 2x + 5| < C$, then $C|x-2| < \varepsilon$, $|x - 2| < \frac{\varepsilon}{C}$

How to find C

If we wish for $\boxed{\delta \le 1}$

$x = 1 \to x^2 + 2x + 5 = 8$

$x = 3 \to x^2 + 2x + 5 = 20$

$|x - 2| < 1$

$-1 < x - 2 < 1$

$1 < x < 3$

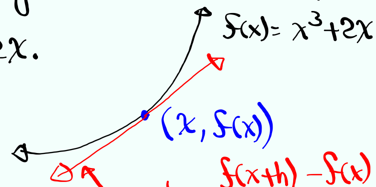
$8 < x^2 + 2x + 5 < 20$

$|x^2 + 2x + 5| < 20$

$\delta = \min\left\{1, \frac{\varepsilon}{20}\right\}$

Sep 17-7:40 AM

Find a function for the slope of a tan. line at any point on the graph of $f(x) = x^3 + 2x$.



$m = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$m = \lim\limits_{h \to 0} \dfrac{(x+h)^3 + 2(x+h) - x^3 - 2x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{2x} + 2h - \cancel{x^3} - \cancel{2x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(3x^2 + 3xh + h^2 + 2)}{\cancel{h}} = \lim\limits_{h \to 0} \left[3x^2 + \cancel{3xh}^{\to 0} + \cancel{h^2}^{\to 0} + 2\right]$

$= \boxed{3x^2 + 2}$

Sep 17-7:52 AM

Working with trig. functions:

$\boxed{\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1}$ & $\boxed{\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} = 0}$

ex: Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = \dfrac{\sin 0}{0} = \dfrac{0}{0}$ I.F.

Method I $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = \lim\limits_{x \to} \dfrac{2 \sin x \cos x}{x}$

$= \lim\limits_{x \to 0} \left[2 \cdot \cos x \cdot \dfrac{\sin x}{x}\right]$

$= \lim\limits_{x \to 0} 2 \cdot \lim\limits_{x \to 0} \cos x \cdot \lim\limits_{x \to 0} \dfrac{\sin x}{x}$

$= 2 \cdot \cancel{\cos 0}^{\,1} \cdot 1$

$= 2 \cdot 1 \cdot 1 = \boxed{2}$

Sep 17-7:59 AM

Method II:

$$\lim_{x\to 0}\frac{2\sin 2x}{2x} = 2\boxed{\lim_{x\to 0}\frac{\sin 2x}{2x}}$$

$$= 2\cdot 1 = \boxed{2}$$

Sep 17-8:05 AM

Evaluate $\displaystyle\lim_{x\to 1}\frac{\sin(x-1)}{x^2-1} = \frac{\sin(1-1)}{1^2-1} = \frac{\sin 0}{0} = \frac{0}{0}$

I.F.

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2-1} = \lim_{x\to 1}\frac{\sin(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x\to 1}\left[\frac{1}{x+1}\cdot\frac{\sin(x-1)}{x-1}\right]$$

$$= \lim_{x\to 1}\frac{1}{x+1}\cdot\boxed{\lim_{x\to 1}\frac{\sin(x-1)}{x-1}}$$

$$= \frac{1}{1+1}\cdot 1 = \boxed{\frac{1}{2}}$$

Sep 17-8:07 AM

Evaluate $\lim\limits_{h\to 0} \dfrac{\sin 3h}{\sin 5h} = \dfrac{\sin 3(0)}{\sin 5(0)} = \dfrac{\sin 0}{\sin 0} = \dfrac{0}{0}$ I.F.

$$\lim_{h\to 0} \frac{\sin 3h}{\sin 5h} = \lim_{h\to 0} \frac{3\,\frac{\sin 3h}{3h}}{5\,\frac{\sin 5h}{5h}}$$

$$= \frac{3}{5} \cdot \frac{\lim\limits_{h\to 0} \frac{\sin 3h}{3h}}{\lim\limits_{h\to 0} \frac{\sin 5h}{5h}} = \frac{3}{5} \cdot \frac{1}{1} = \boxed{\frac{3}{5}}$$

$= \boxed{.6}$

Let $h = .001$

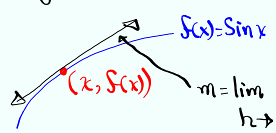
$$\frac{\sin 3h}{\sin 5h} = \frac{\sin(.003)}{\sin(.005)} \approx .60000002$$

Let $h = .0001$

$$\frac{\sin 3h}{\sin 5h} = \frac{\sin(.0003)}{\sin(.0005)} \approx .6$$

Sep 17-8:14 AM

Find a $\boxed{\text{function for slope}}$ of tan. line at $\boxed{\text{any point}}$ on the graph of $\boxed{f(x) = \sin x}$



$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$$m = \lim_{h\to 0} \frac{\sin(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin x[\cos h - 1]}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h\to 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$\lim\limits_{h\to 0} \dfrac{\cos h - 1}{h} \to 0$, $\lim\limits_{h\to 0} \dfrac{\sin h}{h} \to 1$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \boxed{\cos x} \checkmark$$

Sep 17-8:22 AM